

COMPACT HYPER-KÄHLER CATEGORIES I : THEORY

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Abstract

We define the notion of compact hyper-Kähler categories. We study elementary properties of such categories : construction techniques and deformation theory.

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1 Introduction

1.1 Background

An irreducible holomorphically symplectic manifold (or compact hyper-Kähler manifold) is a smooth compact simply connected Kähler manifold with a unique (up to scalar) nowhere degenerate holomorphic 2-form. Together with complex tori and Calabi-Yau manifolds, such varieties are building blocks for the decomposition of Ricci-flat manifolds (see [Bog74, Bea83]). However, contrary to the former two, it is quite hard to produce many different examples of compact hyperkähler manifolds. Up to deformation, the following are the only known examples of such manifolds:

- $S^{[n]}$, the Hilbert-Douady scheme of n points on a $K3$ surface (see [Bea83]),
- $K_n(A)$, the generalized Kummer variety of level n associated to an abelian surface (see [Bea83]),
- A crepant resolution of $\mathcal{M}_S(2, 0, 4)$, the moduli space of semi-stable rank-2 torsion free sheaves with $c_1 = 0$ and $c_2 = 4$ on a $K3$ surface (see [O’G99]),
- A construction similar to the previous one, but where the $K3$ surface is replaced by an abelian surface (see [O’G03]).

On the other hand, the derived categories of compact hyper-Kähler manifolds form an extremely interesting playground to test Kontsevich’s Homological Mirror Symmetry conjecture. Indeed, one expects that such categories have a lot of autoequivalences which do not come from automorphisms of the complex structure but are instead related to Dehn twists along lagrangian projective spaces in the mirror manifold (see [HT06]).¹ Hence, it

¹One also expects the mirror of a hyperkähler manifold to be a twistor deformation of itself (see [Ver99, Huy04]). Combined with Kontsevich’s HMS conjecture, this should give rather strong constraints on the derived category of a compact hyper-Kähler manifold.

seems of high importance to have more examples of derived categories of compact hyper-Kähler manifolds. Or perhaps not so much derived categories of compact hyper-Kähler manifolds at such, but we definitively need more examples of triangulated categories which closely look like these derived categories.

The purpose of this paper and its sequel [Abub] is to introduce the notion of *compact hyper-Kähler categories*, to study some of their basic properties and to provide some interesting new examples. Roughly speaking, a compact hyper-Kähler category of dimension $2n$ is a smooth compact simply connected category with a Serre functor given by the translation by $[2n]$ and endowed with a unique (modulo scalar) non-degenerate categorical 2-form (all these notions will be made precise in the sections 2 and 3 of this paper).

One easily shows that if X is an algebraic variety then $D^b(X)$ is a compact hyper-Kähler category if and only if X is a compact hyperkähler manifold. Our main technique to construct new examples of such categories is based on Kuznetsov's theory of categorical crepant resolution of singularities (see [Kuz08b]). It is well known that one can produce a lot of singular holomorphically symplectic varieties (see [Muk84]) and that crepant resolutions of such varieties are holomorphically symplectic manifolds. Unfortunately, experience tells us that it is almost always impossible to find crepant resolutions of interesting singular holomorphically symplectic varieties (see [CK07, CK06, KL07, LS06, KLS06, MT07, Sac13]).

It is however not too difficult to produce *categorical strongly crepant resolutions of singularities* of some nice singular holomorphically symplectic varieties. For instance, if X is a hyper-Kähler manifold and G is a finite group of automorphisms of X preserving the symplectic form, then X/G admits a categorical strongly crepant resolution of singularities (see Theorem 2.3.1 below for a more general statement). The existence of categorical crepant resolution for all quotient singularities is in clear contrast with the known results in the commutative world. It is indeed a notoriously difficult problem to decide when a quotient singularity of dimension bigger than 4 admits a crepant resolution [Kal02, BKR01].

Hence, it seems that examples of compact hyper-Kähler spaces are way easier to construct in the non-commutative setting. Given the scarcity of examples of commutative hyper-Kähler manifolds, the ease with which one can construct non-commutative incarnations of such varieties plainly justifies, in my opinion, the detailed study of such *hyper-Kähler categories*. Furthermore, unexpected properties of such categories will probably be discovered in the near future and they will certainly shed a new light on the algebraic study of compact hyper-Kähler manifolds.

1.2 Overview of the paper

Let me give a quick overview of the theory of compact hyper-Kähler categories developed in this paper. First of all one would like to give a definition of compact hyper-Kähler categories which is invariant by equivalence. The work of Huybrechts and Nipser-Wisskirchen [HNW11] suggests that it is possible in some geometric cases. Indeed, they prove that if X_1 and X_2 are derived equivalent smooth projective varieties, then X_1 is hyper-Kähler if and only if X_2 is hyper-Kähler. A complete definition of compact hyper-Kähler categories

will be given in section 3 of this paper. For now, let me focus on an important special case:

Definition 1.2.1 *Let X be a smooth projective variety and $\mathcal{T} \subset \mathrm{D}^b(X)$ be a full admissible subcategory and assume furthermore that $\mathcal{O}_X \in \mathcal{T}$. The category \mathcal{T} is said to be **compact hyper-Kähler of dimension $2m$** if the Serre functor of \mathcal{T} is the shift by $2m$ and $H^\bullet(\mathcal{O}_X) = \mathbb{C}[t]/t^{m+1}$, with t homogeneous of degree 2.*

This definition would be invariant (i.e. independent of the embedding) if one could prove that for all smooth projective Y such that \mathcal{T} is a full admissible subcategory of $\mathrm{D}^b(Y)$ containing \mathcal{O}_Y , one has $H^\bullet(\mathcal{O}_Y) \simeq H^\bullet(\mathcal{O}_X)$ (as graded algebras). This does not seem to be easy. Indeed, even in the case where $\mathcal{T} \simeq \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ and X is hyper-Kähler, the proof given in [HNW11] that $H^\bullet(\mathcal{O}_Y) \simeq H^\bullet(\mathcal{O}_X)$ relies on deep structural results for the Hochschild cohomology of compact hyper-Kähler manifolds. Nevertheless, one would expect that these two graded algebras are isomorphic whenever X and Y are derived equivalent. I discuss this invariance problem in more details in [Abuc].

Given this definition, it is easy to check that X is a hyper-Kähler manifold if and only if $\mathrm{D}^b(X)$ is a compact hyper-Kähler category. Of course, one would like to construct new examples of such categories. In section 2 (see Theorem 3.2.4), we will prove:

Theorem 1.2.2 *Let Y be a projective manifold with Gorenstein rational singularities of dimension $2m$. Assume that $\omega_Y \simeq \mathcal{O}_Y$ and that $H^\bullet(\mathcal{O}_Y) \simeq \mathbb{C}[t]/(t^{m+1})$, with t homogenous of degree 2. Any categorical strongly crepant resolution of Y is a compact hyper-Kähler category.*

This result provides us with a whole heap of examples of compact hyper-Kähler categories which are not deformation equivalent one to another. Indeed, let X be a hyper-Kähler manifold and let G_1 and G_2 be two finite groups of automorphisms of X preserving the symplectic form. Then, one proves (see Corollary 3.2.5) that $\mathrm{D}^b(\mathrm{Coh}^{G_1} X)$ and $\mathrm{D}^b(\mathrm{Coh}^{G_2} X)$ are compact hyper-Kähler categories, where $\mathrm{Coh}^G(X)$ is the category of G -equivariant sheaves on X . If G_1 and G_2 are chosen in a astute way, the orbifold topological Euler characteristic of X/G_1 is different from that of X/G_2 . We deduce that $\mathrm{D}^b(\mathrm{Coh}^{G_1} X)$ and $\mathrm{D}^b(\mathrm{Coh}^{G_2} X)$ are not deformation equivalent.

Since the theory of compact hyper-Kähler spaces is based on examples which are often considered up to deformation equivalence, I would like to discuss some properties of smooth deformations of triangulated categories of geometric origin. For example, it is often stated that Hochschild homology numbers of triangulated categories are invariant under deformation. I was unable to find a reference in the literature for such a general statement (see however [Kal09] for some results on cyclic homology). Hence, I think it is worth providing a setting for which we can prove that the Hochschild homology numbers are indeed invariant under deformation. This will be done in the subsection 3.2.

After introducing this context for deformation theory of triangulated categories of geometric origin and proving invariance of Hochschild homology for smooth deformations, I go on studying more precisely deformations of compact hyper-Kähler categories. The following proposition and its proof are straightforward generalizations of the corresponding statement and proof for commutative compact hyper-Kähler manifolds:

Proposition 1.2.3 *Let X be a smooth projective variety, let $\mathcal{T} \subset D^b(X)$ be a full admissible subcategory and B a smooth algebraic variety. Let \mathcal{D} be a smooth deformation of \mathcal{T} over B . Assume that $\mathcal{O}_X \in \mathcal{D}$ and that \mathcal{T} is a compact hyper-Kähler category. Then, there exists a neighborhood $0 \in U \subset B$, such that \mathcal{D}_b is compact hyper-Kähler for all $b \in U$.*

As far as “long-time deformations” are concerned, one can show that a limit of (commutative) compact hyper-Kähler manifolds is again compact hyper-Kähler, provided it is projective. This result is not obvious and requires holonomy techniques to be proved. One would like to know if this holds true in the non-commutative world. I believe that such a deformation result would be important for the theory of compact hyper-Kähler categories and that it can not be proved without the design of new (probably powerful) geometrico-categorical techniques.

Conjecture 1.2.4 (see **Conjecture 3.3.6**) *Let X be a smooth projective variety, let $\mathcal{T} \subset D^b(X)$ be a full admissible subcategory and B a smooth algebraic variety. Let \mathcal{D} be a deformation of \mathcal{T} over B . Assume that \mathcal{D}_b is compact hyper-Kähler for all $b \neq 0$. Then, the category $\mathcal{D}_0 = \mathcal{T}$ is compact hyper-Kähler.*

The first step toward such a result should establish that a deformation of a Calabi-Yau category is again a Calabi-Yau category. This already happens to be not obvious. If one restricts to deformations of Calabi-Yau categories, one can demonstrate the following (see Proposition 3.3.9):

Proposition 1.2.5 *Let X be a smooth projective variety, let $\mathcal{T} \subset D^b(X)$ be a full admissible subcategory which is Calabi-Yau of dimension 4 and let B a smooth algebraic variety. Let \mathcal{D} be a smooth deformation of \mathcal{T} over B with respect to $\pi : \mathcal{X} \rightarrow B$. Assume that $\mathcal{O}_X \in \mathcal{D}$ and that for all $b \neq 0$, the category \mathcal{D}_b is compact hyper-Kähler of dimension 4. Then, the category $\mathcal{D}_0 = \mathcal{T}$ is compact hyper-Kähler of dimension 4.*

It would certainly be desirable to know if this result can be generalized to higher-dimensional cases. It would also be very interesting to discover which structural results known for the Hochschild cohomology rings of compact hyper-Kähler manifolds are still valid in the categorical context. For instance, if X is a compact hyper-Kähler manifold of dimension $2m$ and $\mathrm{HH}^{(2)}(X)$ is the sub-algebra of $\mathrm{HH}^*(X)$ generated by $\mathrm{HH}^2(X)$, Verbitsky [Ver96] proved the following isomorphism:

$$\mathrm{HH}^{(2)}(X) \simeq S^* \mathrm{HH}^2(X) / \{a^{m+1}, \text{ such that } q(a) = 0\},$$

where q is the Beauville-Bogomolov quadratic form. This result is heavily used in [HNW11] to prove the derived invariance of the compact hyper-Kähler property for projective varieties. In my opinion, it would be fascinating to have a similar statement for the Hochschild cohomology of a compact hyper-Kähler category.

Conjecture 1.2.6 *Let \mathcal{T} be a compact hyper-Kähler category of dimension $2m$. There exists a non-degenerate quadratic form q on $\mathrm{HH}^2(\mathcal{T})$ and an isomorphism:*

$$\mathrm{HH}^{(2)}(\mathcal{T}) \simeq S^* \mathrm{HH}^2(\mathcal{T}) / \{a^{m+1}, \text{ such that } q(a) = 0\},$$

where $\mathrm{HH}^{(2)}(\mathcal{T})$ is the sub-algebra of $\mathrm{HH}^*(\mathcal{T})$ generated by $\mathrm{HH}^2(\mathcal{T})$.

I will not discuss any such cohomological result in the present paper. In its sequel [Abub], I will work out in details one modular example of dimension 4, its connections with the theory of categorical fixed point loci and non-commutative compactifications of moduli space of Sp_{2n} Higgs bundles. In particular, using some computations that Grégoire Menet provided me with, I will show that the Hochschild cohomology ring of this non-commutative compact hyper-Kähler manifold shares many properties with those of classical commutative compact hyper-Kähler manifolds.

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2 Categorical crepant resolution of singularities

As mentioned in the introduction, our examples of compact hyper-Kähler categories are based on the theory of categorical crepant resolutions of singularities. This notion has been developed in [Kuz08b] and was further explored in [Abu13a, Abu13b].

2.1 Definition and motivations

Let us recall that a *crepant* resolution of a normal Gorenstein algebraic variety Y is a resolution of singularities $\pi : X \rightarrow Y$ such that $\pi^*\omega_Y = \omega_X$, where ω_Y is the dualizing line bundle of Y . Crepant resolutions are often considered to be minimal resolutions of singularities (see the first part of [Abu13a] for an extended discussion about minimality for resolutions of singularities). Unfortunately crepant resolutions of singularities are quite rare. The following example is very classical:

Example 2.1.1 *Let Y be a cone over $v_2(\mathbb{P}^3) \subset \mathbb{P}(S^2\mathbb{C}^4)$. The variety Y is analytically equivalent to $\mathbb{C}^6/\{1, -1\}$. Hence, it is locally analytically \mathbb{Q} -factorial (see [KM98], Chapter 5), so that it has no small resolution of singularities. Furthermore the blow-up of Y along the vertex gives a resolution of singularities where the coefficient of the exceptional divisor in the dualizing bundle formula is 1 (this is an obvious computation). As a consequence, the variety Y has terminal singularities. Since it admits no small resolution, we find that Y has no crepant resolution of singularities.*

Given a singularity which does not admit any crepant resolution, one still would like to know if it is possible to produce minimal resolutions from the point of view of category theory. Kuznetsov's insight is that such categorical "minimal" resolutions should be constructed as *categorical crepant resolution* (see [Kuz08b], section 4).

Definition 2.1.2 *Let Y be an algebraic variety and \mathcal{X} be a smooth Deligne-Mumford stack. We say that \mathcal{X} **homologically dominates** Y , if there exists a proper morphism $p : \mathcal{X} \rightarrow Y$, such that $Rp_*\mathcal{O}_{\mathcal{X}} \simeq \mathcal{O}_Y$.*

Typical examples of such phenomenon include resolutions of singularities for a variety with rational singularities and the canonical projection from a smooth Deligne-Mumford stack to its coarse moduli space.

Definition 2.1.3 *Let Y be an algebraic variety with Gorenstein rational singularities. Let $p : \mathcal{X} \rightarrow Y$ be a smooth Deligne-Mumford stack which homologically dominates Y . A **categorical resolution** of Y is a full admissible subcategory $\mathcal{T} \subset \mathrm{D}^b(\mathcal{X})$ such that $\mathbf{L}p^* \mathrm{D}^{\mathrm{perf}}(Y) \subset \mathcal{T}$.*

In [Kuz08b], the definition of categorical resolution was restricted to the case where X is a variety. A way more general notion of categorical resolution has been defined and studied by Kuznetsov and Lunts in [KL12]. The main advantage of their definition is that one can prove the existence of a categorical resolution for any scheme (!) of finite type over \mathbb{C} . With Definition 2.1.3, we lie in the middle. The possibility to work with Deligne-Mumford stacks allows to produce interesting examples of non-commutative resolution of singularities (see [Abua]). On the other hand, many elementary techniques and results from [Kuz08b] are still valid when X is a smooth Deligne-Mumford stack, with proofs being exactly the same.

Definition 2.1.4 (Categorical crepancy, [Kuz08b]) *Let Y be an algebraic variety with Gorenstein rational singularities and $p : \mathcal{X} \rightarrow Y$ be a Deligne-Mumford stack homologically dominating Y . Let $\delta : \mathcal{T} \hookrightarrow \mathrm{D}^b(\mathcal{X})$ be a categorical resolution of Y and let $p_{\mathcal{T}*} : \mathcal{T} \rightarrow \mathrm{D}^b(Y)$ be the composition of $\mathbf{R}p_*$ with δ .*

- We say that $p_{\mathcal{T}*} : \mathcal{T} \rightarrow \mathrm{D}^b(Y)$ is a **weakly crepant resolution** of Y , if for all $\mathcal{F} \in \mathrm{D}^{\mathrm{perf}}(Y)$, we have:

$$p_{\mathcal{T}}^* \mathcal{F} \simeq p_{\mathcal{T}}^! \mathcal{F},$$

where $p_{\mathcal{T}}^*$ and $p_{\mathcal{T}}^!$ are the left and right adjoint to $p_{\mathcal{T}*}$.

- We say that $p_{\mathcal{T}*} : \mathcal{T} \rightarrow \mathrm{D}^b(Y)$ is a **strongly crepant resolution** of Y if the following two conditions hold:

1. we have $\mathbf{L}p^* \mathcal{F} \otimes_{\mathcal{O}_X} \delta \mathcal{G} \in \mathcal{T}$, for all $\mathcal{F} \in \mathrm{D}^{\mathrm{perf}}(Y)$ and $\mathcal{G} \in \mathcal{T}$,
2. the identity functor is a relative Serre functor for \mathcal{T} with respect to the map $p_{\mathcal{T}*}$.

Let us make a few comments on this definition. The first requirement in the definition of a categorical strongly crepant resolution is that \mathcal{T} has a module structure over $\mathrm{D}^{\mathrm{perf}}(Y)$ (see [Kuz08b], section 3). Assume that \mathcal{T} is a categorical strongly crepant resolution of a projective variety Y . Then the (absolute) Serre functor of \mathcal{T} is given by the tensor product by $\pi^* \omega_Y[\dim Y]$. Note also that a categorical strongly crepant resolution of a Gorenstein rational singularity is automatically a categorical weakly crepant resolution of this singularity, but the converse is not true (see [Kuz08b], section 8). However, in the purely geometric setting (that is when $\mathcal{T} \simeq \mathrm{D}^b(X)$, for some algebraic variety X), all these notions coincide. Indeed we have the:

Proposition 2.1.5 (see [Abu13a]) *Let $p : X \rightarrow Y$ be a morphism of algebraic varieties. Then we have the equivalences:*

$\mathbf{R}p_* : \mathrm{D}^b(X) \rightarrow \mathrm{D}^b(Y)$ *is a categorical strongly crepant resolution of singularities*

$$\Longleftrightarrow$$

$Rp_* : D^b(X) \rightarrow D^b(Y)$ is a categorical weakly crepant resolution of singularities

$$\Longleftrightarrow$$

$p : X \rightarrow Y$ is a crepant resolution of singularities.

This proposition implies that Kuznetsov's definition is well-formed. Indeed, in the purely geometrical setting, there is no difference whatsoever between crepant resolutions, categorical weakly crepant resolutions and categorical strongly crepant resolutions.

However, if X is a variety without any crepant resolution of singularities, it seems much easier to construct a weakly crepant resolution categorical resolution of X . Indeed, we have the following:

Theorem 2.1.6 ([Abu13b]) *All Gorenstein determinantal varieties (square, symmetric or pfaffian) admit weakly crepant resolution of singularities.*

It is well-known that pfaffian varieties do not have any geometric crepant resolution of singularities ², see [Abu13a], appendix B for instance. As a consequence, this result sheds a new light on what could possibly be a minimal categorical resolution for determinantal varieties. Of course, one would like to know when determinantal varieties have categorical strongly crepant resolutions of singularities.

Let us focus on a few *basic* local type of singularities ³ for which we know how to construct a categorical strongly crepant resolution of singularities:

Theorem 2.1.7 ([Kuz08b], [Kuz08a]) *The following local type of singularities do not have any geometric crepant resolution of singularities but they admit categorical strongly crepant resolutions of singularities:*

- a cone over the d -th Veronese embedding $v_d(\mathbb{P}^n) \subset \mathbb{P}(S^d \mathbb{C}^{n+1})$, when d divides $n+1$,
- a cone over the Grassmannian $G(2, V)$ in its Plücker embedding for $\dim V$ odd,
- The Pfaffian $\mathbb{P}f_4(V) := \mathbb{P}\{v \in \bigwedge^2 V, |\mathrm{rk} v| \leq 4\}$ for $\dim V$ odd.

As mentioned above, we do expect that the Pfaffians $\mathbb{P}f_k(V) := \mathbb{P}\{v \in \bigwedge^2 V, |\mathrm{rk} v| \leq k\}$ admits a categorical strongly crepant resolution if and only if $\dim V$ is odd. This contrasts with the fact that $\mathbb{P}f_k(V)$ admits always a categorical weakly crepant resolution.

Note that the first two examples above fit in the same picture : they are all cones over a Fano varieties which admit *rectangular Lefschetz decompositions* with respect to their embedded polarizations.

²It is also the case for symmetric determinantal varieties, except for the hypersurface case, [Abu13a], appendix B.

³By basic, I mean that this singularity is not a product of a smaller dimensional singularity by a smooth variety.

2.2 Lefschetz decompositions and applications

In this subsection, we will define the notion of Lefschetz decomposition and state a result of Kuznetsov which relates rectangular Lefschetz decompositions and categorical strongly crepant resolutions of singularities.

Definition 2.2.1 *Let X be an algebraic variety and let L be a line bundle over X . A **Lefschetz decomposition of X with respect to L** is a decomposition:*

$$\mathrm{D}^b(X) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes L, \dots, \mathcal{A}_m \otimes L^{\otimes m} \rangle,$$

where $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_m$ are admissible subcategories of $\mathrm{D}^b(X)$ such that $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots \supset \mathcal{A}_m$.

The decomposition is furthermore said to be **rectangular** if $\mathcal{A}_0 = \mathcal{A}_1 = \dots = \mathcal{A}_m$.

We refer to [Kuz08b, Kuz07] for more details about Lefschetz decompositions. Let us mention that \mathbb{P}^n admits a rectangular Lefschetz decomposition with respect to $\mathcal{O}_{\mathbb{P}^n}(d)$ if and only if d divides $n + 1$:

$$\mathrm{D}^b(\mathbb{P}^n) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{O}_{\mathbb{P}^n}(d), \dots, \mathcal{A}_{\frac{n+1}{d}-1} \otimes \mathcal{O}_{\mathbb{P}^n}(n + 1 - d) \rangle,$$

where $\mathcal{A}_0 = \mathcal{A}_1 = \dots = \mathcal{A}_{\frac{n+1}{d}-1} = \langle \mathcal{O}_{\mathbb{P}^n}, \dots, \mathcal{O}_{\mathbb{P}^n}(d - 1) \rangle$. In the same vein, the Grassmannian $G(2, V)$ admits a rectangular Lefschetz decomposition with respect to $\mathcal{O}_{G(2, V)}(1)$ if and only if $\dim V$ is odd:

$$\mathrm{D}^b(G(2, V)) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{O}_{G(2, V)}(1), \dots, \mathcal{A}_{\dim V - 1} \otimes \mathcal{O}_{G(2, V)}(\dim V - 1) \rangle,$$

with $\mathcal{A}_0 = \mathcal{A}_1 = \dots = \mathcal{A}_{\dim V - 1} = \langle \mathcal{O}_{G(2, V)}, \mathcal{U} \rangle$ where \mathcal{U} is the tautological line bundle on $G(2, V)$.

The following theorem makes the link between rectangular Lefschetz decompositions and strongly crepant resolutions of singularities:

Theorem 2.2.2 ([Kuz08b]) *Let Y be an algebraic variety with Gorenstein and rational singularities. Let $p : X \rightarrow Y$ be a resolution of singularities with exceptional divisor E such that:*

- *there exists an integer m such that $\omega_X = \pi^* \omega_Y \otimes \mathcal{O}_E(mE)$,*
- *E admits a rectangular Lefschetz decomposition with respect to $\mathcal{O}_E(-E)$:*

$$\mathrm{D}^b(E) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{O}_E(-E), \dots, \mathcal{A}_m \otimes \mathcal{O}_E(-mE) \rangle,$$

with $Lp^* \mathrm{D}^{\mathrm{perf}}(p(E)) \subset \mathcal{A}_0$ and \mathcal{A}_0 is stable with respect to tensoring with elements of $Lp^* \mathrm{D}^{\mathrm{perf}}(p(E))$.

Then Y admits a categorical strongly crepant resolution of singularities.

Note that the first hypothesis in the theorem is automatically satisfied if E is irreducible. On the other hand, we need an extremely precise description of the fibers of the map $p : E \rightarrow p(E)$ to see if the second hypothesis holds. It is quite hard in practice to check the second hypothesis. If one is interested only in categorical weakly crepant resolution of singularities, then more general results have been obtained in [Abu13b].

In the following, we apply directly Kuznetsov's theorem in order to get a concrete example of categorical strongly crepant resolution of singularities. It plays a central role in [Abub].

Proposition 2.2.3 *Let Y be an algebraic variety with isolated singular points all analytically equivalent to a cone over $v_2(\mathbb{P}^3) \subset \mathbb{P}(S^2\mathbb{C}^4)$. There exists a categorical strongly crepant resolution of singularities for Y .*

Thanks to example 2.1.1, a variety as in Proposition 2.2.3 does not admit any geometric crepant resolution of singularities.

Proof:

► Let y_1, \dots, y_p be the singular points of Y and let $\pi : X \rightarrow Y$ be the blow-up of Y along y_1, \dots, y_p . The exceptional divisor E has p connected components E_1, \dots, E_p which are all isomorphic to \mathbb{P}^3 . Moreover, for all $i = 1 \dots p$ we have $E|_{E_i} = \mathcal{O}_{\mathbb{P}^3}(2)$.

As a consequence, we have a semi-orthogonal decomposition of E :

$$\mathrm{D}^b(E) = \langle \mathcal{A}_0, \mathcal{A}_1 \otimes \mathcal{O}_E(-E) \rangle,$$

with $\mathcal{A}_0 = \mathcal{A}_1 = \langle \mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_E(-\frac{1}{2}(E)) \rangle$ and $\mathcal{O}_E(-\frac{1}{2}(E))$ is the unique line bundle on E which restricts to $\mathcal{O}_{\mathbb{P}^3}(1)$ on each E_i . One can now apply Theorem 2.2.2, since \mathcal{A}_0 is obviously stable when tensoring by elements of $\mathrm{L}\pi^*\mathrm{D}^{\mathrm{perf}}(\{y_1, \dots, y_p\})$. ◀

2.3 Categorical crepant resolutions and quotient singularities

In this sub-section, I will recall the main result of [Abua]. It will be useful to construct new compact hyper-Kähler categories starting from a compact hyper-Kähler variety endowed with a finite group of symplectic automorphisms.

Theorem 2.3.1 ([Abua]) *Let X be a quasi-projective variety with normal Gorenstein quotient singularities and let \mathcal{X} be a smooth separated Deligne-Mumford stack whose coarse moduli space is X . Assume that the dualizing line bundle of \mathcal{X} is the pull back of the dualizing line bundle on X , then $\mathrm{D}^b(\mathcal{X})$ is a strongly crepant resolution of X .*

Furthermore, there exists a sheaf of algebras \mathcal{A} on X such that $\mathrm{D}^b(\mathcal{X}) \simeq \mathrm{D}^b(X, \mathcal{A})$. Hence, the pair (X, \mathcal{A}) is a non-commutative crepant resolution of X in the sense of Van den Bergh.

Note that if X is a normal quasi-projective variety with quotient singularities, there is always a smooth separated Deligne-Mumford stack associated to it as in the above statement (see Proposition 2.8 of [Vis89]). The non-trivial hypothesis (which can not be removed) is that the dualizing bundle of the Deligne-Mumford stack associated to X is the pull back of the dualizing bundle on X . This amounts to check that on an étale atlas of \mathcal{X} , the line bundle $\omega_{\mathcal{X}}$ is equivariantly ⁴ locally trivial. This finally boils down to checking that for any $x \in X$, there exists an étale neighborhood U_x of $x \in X$, such that $U_x = V/G$ where V is a vector space and G is a subgroup of $\mathrm{SL}(V)$. This holds in particular for a variety whose singularities are isolated points locally analytically equivalent to a cone over $v_2(\mathbb{P}^3) \subset \mathbb{P}(S^2\mathbb{C}^4)$. In that case, the crepant resolution produced by theorem 2.3.1 matches with the one described in proposition 2.2.3 (this is proved in the last section of [Abua]).

In the local case, the above result was already known for a long time (see [vdB04], for instance). The main point of Theorem 2.3.1 is to prove the existence result of categorical

⁴for the isotropy groups of the fixed points of the étale atlas of \mathcal{X}

crepant resolutions for quotient singularities in the **global setting**. Indeed, there is a priori no reason for the local resolutions constructed in [vdB04] to glue globally. The main point of [Abua] is to exhibit a sheaf of non-commutative algebras which provides such a gluing of the local resolutions.

3 Compact hyper-Kähler categories

Recall that a holomorphically symplectic variety of dimension $2m$ is (in the projective case) a smooth projective variety X having trivial canonical bundle and endowed with a 2-form $\sigma \in H^0(X, \Omega_X^2)$, such that $\sigma^{\wedge m} \neq 0$. One says that X is compact hyper-Kähler if X is simply connected and σ generates $H^0(X, \Omega_X^2)$. Since σ defines an isomorphism $\sigma : \Omega_X \rightarrow T_X$, one can equivalently say that X is holomorphically symplectic if X is smooth simply connected projective with trivial canonical bundle and there exists a Poisson bracket $\theta \in H^0(X, \bigwedge^2 T_X)$, such that $\theta^{\wedge m} \neq 0$. Hence, one could be tempted to give the following definition:

Definition 3.0.2 (naive definition of holomorphically symplectic categories) *Let \mathcal{T} be a smooth compact triangulated category. We say that \mathcal{T} is **holomorphically symplectic of dimension $2m$** if the shift by $2m$ is a Serre functor for \mathcal{T} and there exists $\theta \in \mathrm{HH}^2(\mathcal{T})$ such that $\theta^{\mathrm{om}} \neq 0$.*

Such a definition has the advantage to be invariant by equivalences. Its (non-negligible) drawback is that the derived categories of many non holomorphically symplectic varieties are then to be considered as holomorphically symplectic categories. Indeed, the Hochschild-Kostant-Rosenberg isomorphism [Mar09] shows that for X smooth projective, there is a decomposition (compatible with products on both sides [CRVdB12, HNW11]):

$$\mathrm{HH}^2(X) = H^0(X, \bigwedge^2 T_X) \oplus H^1(X, T_X) \oplus H^2(X, \mathcal{O}_X).$$

Hence, with Definition 3.0.2, the derived category of an abelian surface would be considered as a holomorphically symplectic category, which is something we want to avoid. The main problem here is to define categorically one of the algebras $H^0(X, \bigwedge^* T_X)$ or $H^0(X, \bigwedge^* \Omega_X)$ (the latter being isomorphic, by Hodge duality, to the algebra $H^*(X, \mathcal{O}_X)$). We will try to give such a definition in the first subsection below.

3.1 Homological units

Let X be an algebraic variety and let $\mathcal{F} \in \mathrm{D}^b(X)$ be an object whose rank is not zero. Then the trace map:

$$\mathrm{Tr} : \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{F}) \rightarrow \mathcal{O}_X$$

splits and gives a splitting:

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{F}) = H^\bullet(\mathcal{O}_X) \oplus \mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{F})_0,$$

where $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{F})_0$ is the graded vector space of trace-less endomorphisms. Hence, the algebra $H^\bullet(\mathcal{O}_X)$ appears as a maximal direct factor of the endomorphisms algebra of any object in $\mathrm{D}^b(X)$ whose rank is not vanishing. We will see below that this algebra is

an important categorical invariant and I believe it is worth studying it (categorically) in more details and in greater generality.

Definition 3.1.1 *Let \mathcal{C} be an abelian category with a non-trivial rank function and \mathcal{T} be an admissible subcategory in $D^b(\mathcal{C})$. A graded algebra \mathfrak{T}^\bullet is called an homological unit for \mathcal{T} (with respect to \mathcal{C}), if \mathfrak{T}^\bullet is maximal for the following properties :*

1. *for any object $\mathcal{F} \in \mathcal{T}$, there exists a graded algebra morphism $\mathfrak{T}^\bullet \rightarrow \text{Hom}^\bullet(\mathcal{F}, \mathcal{F})$ which is functorial in the following sense. Let $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ and let $a \in \mathfrak{T}^k$ for some k . We denote by $a_{\mathcal{F}}$ (resp. $a_{\mathcal{G}}$) the image of a in $\text{Hom}^k(\mathcal{F}, \mathcal{F})$ (resp. $\text{Hom}^k(\mathcal{G}, \mathcal{G})$). Then, for any morphism $\psi : \mathcal{F} \rightarrow \mathcal{G}$, there is a commutative diagram:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{a_{\mathcal{F}}} & \mathcal{F}[k] \\ \downarrow \psi & & \downarrow \psi[k] \\ \mathcal{G} & \xrightarrow{a_{\mathcal{G}}} & \mathcal{G}[k] \end{array}$$

2. *for any $\mathcal{F} \in \mathcal{T}$ which rank (seen as an object in $D^b(\mathcal{C})$) is not vanishing, the morphism $\mathfrak{T}^\bullet \rightarrow \text{Hom}^\bullet(\mathcal{F}, \mathcal{F})$ is an injection of graded algebras, which splits as a morphism of vector spaces.*

With hypotheses as above, an object $\mathcal{F} \in \mathcal{T}$ is said to be unitary, if $\text{Hom}^\bullet(\mathcal{F}, \mathcal{F}) = \mathfrak{T}^\bullet$, where \mathfrak{T}^\bullet is a homological unit for \mathcal{T} .

Of course, one can not expect that all examples of homological units as defined above will be significant. In the main applications of the present paper, one will look at $\mathcal{C} = \text{Coh}(X)$, $\text{Coh}^G(X)$ or $\text{Coh}(X, \alpha)$, where X is a smooth projective variety, G an algebraic group acting on X , α a Brauer class on X and the rank function will be the obvious one. However, it is well possible that many new examples of homological units coming from Representation Theory will be discovered, so that it seems sensible to give a general definition that does not restrict to purely geometrical examples.

Note also that the hypothesis of non-vanishing rank for the splitting is a technical hypothesis which is important. It would be very interesting to know if there are some non-trivial examples where the splitting occurs whatever the rank of the object.

Example 3.1.2 1. *Let X be a smooth algebraic variety and $\alpha \in \text{Br}(X)$, a class in the Brauer group of X . Consider $\mathcal{C} = \text{Coh}(X, \alpha)$, the category of coherent α -twisted sheaves on X . One can define a rank function on \mathcal{C} as being the rank of \mathcal{F} when seen as an \mathcal{O}_X -module. Then for any $\mathcal{F} \in D^b(\mathcal{C})$, we have a trace map:*

$$\text{Tr} : \mathbf{R}\mathcal{H}om_{D^b(\mathcal{C})}(\mathcal{F}, \mathcal{F}) \rightarrow \mathcal{O}_X$$

which splits when the rank of \mathcal{F} is not zero. As a consequence, for all $\mathcal{F} \in D^{\text{perf}}(\mathcal{C})$, we have a graded algebra morphism:

$$H^\bullet(\mathcal{O}_X) \rightarrow \text{Hom}_{D^b(\mathcal{C})}^\bullet(\mathcal{F}, \mathcal{F})$$

which is split (as a morphism of vector spaces) when the rank of \mathcal{F} is not zero. The morphism $H^\bullet(\mathcal{O}_X) \rightarrow \mathrm{Hom}_{\mathrm{D}^b(\mathcal{C})}^\bullet(\mathcal{F}, \mathcal{F})$ is given by $a \rightarrow \mathrm{id}_{\mathcal{F}} \otimes a$, so that the functoriality property is clearly satisfied. Furthermore, if L is a twisted line bundle in $\mathrm{D}^b(\mathrm{Coh}(X, \alpha))$, we have $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{C})}^\bullet(L, L) = H^\bullet(\mathcal{O}_X)$. Thus, $H^\bullet(\mathcal{O}_X)$ is maximal for the properties required in Definition 3.1.1 and it is a homological unit for \mathcal{C} .

2. Let X be a smooth algebraic variety and G be a finite group acting on X . For any $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}^G(X))$, the trace map $\mathrm{Tr} : \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{F}) \rightarrow \mathcal{O}_X$ is G -equivariant and it is split if the rank of \mathcal{F} is non-zero. Hence, for all $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}^G(X))$, we have a graded algebra morphism:

$$H^\bullet(\mathcal{O}_X)^G \rightarrow \mathrm{Hom}_{\mathrm{D}^b(\mathrm{Coh}^G(X))}^\bullet(\mathcal{F}, \mathcal{F}),$$

which is split (as a morphism of vector spaces) when the rank of \mathcal{F} is not zero. The morphism $H^\bullet(\mathcal{O}_X)^G \rightarrow \mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{F})$ is again given by $a \rightarrow \mathrm{id}_{\mathcal{F}} \otimes a$, so that the functoriality property is also satisfied. Furthermore, if L is a G -invariant line bundle on X , we have $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{C})}^\bullet(L, L) = H^\bullet(\mathcal{O}_X)^G$. Hence, the algebra $H^\bullet(\mathcal{O}_X)$ is maximal for the properties required in Definition 3.1.1 and it is a homological unit for \mathcal{C} . This readily generalizes for any smooth Deligne-Mumford stack. Namely, if \mathcal{X} is a smooth Deligne-Mumford stack, then $H^\bullet(\mathcal{O}_{\mathcal{X}})$ is a homological unit for $\mathrm{D}^b(\mathcal{X})$. Note that all line bundles on \mathcal{X} are unitary objects.

One would like to know when the homological unit is unique and independent of the embedding in the derived category of an abelian category. This question seems to be interesting for itself and it does not have an obvious answer. I discuss this invariance problem in [Abuc]), where I give some applications to the conjecture of derived invariance of Hodge numbers.

3.2 Definition and construction techniques

We first recall the definition of smoothness, compactness and regularity for triangulated categories.

Definition 3.2.1 ([Kon09], [Orl14]) *Let \mathcal{T} be the derived category of DG-modules over some DG-algebra (\mathcal{A}, d) (over \mathbb{C}). The category \mathcal{T} is said to be:*

- **smooth**, if \mathcal{A} is a perfect bi-module over $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^{op}$.
- **compact**, if $\dim H^\bullet(\mathcal{A}, d) < +\infty$.
- **regular**, if it has a strong generator.
- **Calabi-Yau of dimension p** if the shift by p is a Serre functor for \mathcal{A} .

Assume that $\mathcal{T} = \mathrm{D}^b(X)$, where X is an algebraic over \mathbb{C} . It is easily shown that X is smooth and proper over \mathbb{C} if and only if \mathcal{T} is smooth and compact (see [Kon09]). Note also that if \mathcal{T} is a semi-orthogonal component of the derived category of a smooth proper scheme over \mathbb{C} , then \mathcal{T} is smooth, compact and regular (see [Orl14]). With these definitions in hand, we can introduce the main notion of this paper:

Definition 3.2.2 (compact hyper-Kähler categories) *Let \mathcal{T} be a smooth, compact and regular triangulated category which is closed under direct summands. Assume that \mathcal{T} is a semi-orthogonal component of $D^b(\mathcal{C})$, where \mathcal{C} is an abelian category with a rank function. We say that \mathcal{T} is a **compact hyper-Kähler category** if \mathcal{T} is Calabi-Yau of dimension $2m$ and there is a unique homological unit for \mathcal{T} (with respect to its embedding in $D^b(\mathcal{C})$), which is isomorphic $\mathbb{C}[t]/(t^{m+1})$ with t homogeneous of degree 2.*

Proposition A.1 of [HNW11] implies that a projective variety X of dimension $2m$ such that $H^\bullet(\mathcal{O}_X) \simeq \mathbb{C}[t]/(t^{m+1})$ is necessarily simply connected. By Hodge duality, we deduce immediately the following:

Proposition 3.2.3 *Let X be an algebraic variety. The category $D^b(X)$ is compact hyper-Kähler if and only if the variety X is compact hyper-Kähler.*

We will see that one can construct many examples of compact hyper-Kähler categories which are non-commutative. It seems extremely hard to find new examples of commutative compact hyper-Kähler manifolds. Actually, one can produce a lot of compact singular holomorphically symplectic varieties [Muk84]. But almost all of them do not admit any geometric crepant resolution of singularities. Hence, I believe that the following result opens the door to a new world of compact hyper-Kähler spaces.

Theorem 3.2.4 *Let Y be a projective manifold with Gorenstein rational singularities of dimension $2m$. Assume that $\omega_Y = \mathcal{O}_Y$ and that $H^\bullet(\mathcal{O}_Y) \simeq \mathbb{C}[t]/(t^{m+1})$, with t homogenous of degree 2. Any categorical strongly crepant resolution of Y is a compact hyper-Kähler category.*

The above statement is slightly ambiguous as we haven't proved that the notion of compact hyper-Kähler category is independent of the embedding inside the derived category of an abelian category with a rank function. However, our definition of categorical resolution always refer to a Deligne-Mumford stack which homologically dominates Y . In the above statement, we implicitly refer to the embedding of \mathcal{T} inside the derived category of this Deligne-Mumford stack.

Proof :

►

Let $p : \mathcal{X} \rightarrow Y$ be a projective Deligne-Mumford stack which homologically dominates Y and let $\mathcal{T} \subset D^b(\mathcal{X})$ be an admissible full subcategory such that the induced map $\mathbf{R}p_* : \mathcal{T} \rightarrow D^b(Y)$ is a strongly crepant resolution. Since \mathcal{T} is an admissible subcategory of the derived category of a smooth projective Deligne-Mumford stack, we know that \mathcal{T} is smooth, compact and regular. Furthermore, it is a strongly crepant resolution of a Gorenstein projective variety whose dualizing bundle is trivial, hence \mathcal{T} is Calabi-Yau of dimension $\dim Y = 2m$.

We are only left to prove that there is a unique homological unit for \mathcal{T} (with respect to its embedding inside $D^b(\mathcal{X})$), which is isomorphic to $\mathbb{C}[t]/(t^{m+1})$ with t homogeneous of degree 2. By hypothesis, we have $\mathbf{R}p_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_Y$, so that $H^\bullet(\mathcal{O}_{\mathcal{X}}) \simeq H^\bullet(\mathcal{O}_Y) \simeq \mathbb{C}[t]/(t^{m+1})$ (with t homogeneous of degree 2) is a homological unit for $D^b(\mathcal{X})$. Hence, for all $\mathcal{F} \in \mathcal{T}$, we have a graded algebra morphism:

$$\mathbb{C}[t]/(t^{m+1}) \rightarrow \mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{F}),$$

given by $a \rightarrow id_{\mathcal{F}} \otimes a$. As a consequence, this morphism satisfies the functoriality condition stated in definition 3.1.1. Furthermore this morphism is split when the rank of \mathcal{F} is not zero. But $\mathcal{O}_X \in \mathcal{T}$, so that there is a unique homological unit for \mathcal{T} (with respect to its embedding in $D^b(\mathcal{X})$), which is isomorphic to $\mathbb{C}[t]/(t^{m+1})$ with t homogeneous of degree 2. \blacktriangleleft

Corollary 3.2.5 *Let X be a compact hyper-Kähler variety and G be a finite group of symplectic automorphisms of X . The category $D^b(Coh^G(X))$ is a compact hyper-Kähler category.*

Proof :

► One can show directly that $D^b(Coh^G(X))$ is a compact hyper-Kähler category, but I think it is interesting to show it is a consequence of Theorem 3.2.4. Indeed, if G is a finite group of symplectic automorphisms of X , then X/G is a projective Gorenstein variety with rational singularities. The generator of $H^2(\mathcal{O}_X)$ being G -equivariant, it descends to X/G and its top wedge-product remains non zero on X/G . As a consequence, we have $H^\bullet(\mathcal{O}_{X/G}) \simeq \mathbb{C}[t]/(t^{m+1})$ with t homogeneous of degree 2 (m is the half-dimension of X). Theorem 2.3.1 implies that $D^b(Coh^G(X))$ is a strongly crepant resolution of X/G . Theorem 3.2.4 then proves that $D^b(Coh^G(X))$ is a compact hyper-Kähler category. \blacktriangleleft

There has been recently quite a bit of work on symplectic automorphisms of compact hyper-Kähler manifolds ([Cam12, BNWS13, Mon13]). Using the existing results in the literature and corollary 3.2.5, one might hope to construct a vast number of different deformation families of compact hyper-Kähler categories in each even dimension. This would put the theory of compact hyper-Kähler categories on the same footing as the theory of strict Calabi-Yau manifolds : we still have no efficient tools to classify them, but one can construct a large amount of non-equivalent (up to deformation) examples of such spaces in each fixed dimension.

Remark 3.2.6 *The notion of (holomorphically) symplectic stack has been defined by Pantev, Toën, Vaquié and Vezzosi [PTVV13] and by Zhang [Zha11]. It would be of course desirable to know if one can define the notion of irreducible holomorphically symplectic stack and if the derived categories of such stacks are related to compact hyper-Kähler categories.*

One of my primary goal when developing the theory of compact hyper-Kähler categories was to understand whether the sporadic examples of compact hyper-Kähler manifolds discovered by O’Grady could be part of a larger sequence of examples living in the non-commutative world. Let $\mathcal{M}_{K3}(2, 0, 2r)$ be the moduli space of rank 2 torsion free sheaves with $c_1 = 0$, $c_2 = 2r$ and which are semi-stable with respect to a generic polarization. Because of the parity of c_2 , these moduli spaces are not smooth for $r \geq 2$. O’Grady proved that $\mathcal{M}_{K3}(2, 0, 4)$ admits a crepant resolution and that this crepant resolution is a compact hyper-Kähler manifold which is not deformation equivalent to the previously known examples of compact hyper-Kähler manifolds [O’G99].

It was then proved in [CK07] and [KLS06] that the moduli spaces $\mathcal{M}_{K3}(2, 0, 2r)$ do not admit any crepant resolution for $r \geq 3$. Hence one can’t hope to find new examples of commutative compact hyper-Kähler variety starting with these moduli spaces. However, it seems quite likely that these moduli spaces have categorical crepant resolutions.

Exhibiting such resolutions would provide a whole new heap of compact hyper-Kähler categories. This would also demonstrate that the O’Grady examples are not sporadic at all : they would be part of a series which naturally lives in the non-commutative world.

Question 3.2.7 *Let $r \geq 3$ be an integer. Does the moduli spaces $\mathcal{M}_{K3}(2, 0, 2r)$ admit a categorical strongly crepant resolution of singularities?*

3.3 Deformation theory for compact hyper-Kähler categories

In this subsection, I will prove some basic results for the deformation theory of compact hyper-Kähler categories. I will use them in [Abub] to prove that there exists non-commutative compact hyper-Kähler fourfolds which are not deformation equivalent to any commutative compact hyper-Kähler fourfold. In particular, if it exists, the moduli space of non-commutative compact hyper-Kähler fourfold contains a component which is purely non-commutative.

I will focus on a specific type of deformation of triangulated categories : deformation inside the derived category of an algebraic variety (all results proven below should carry on without any problem to deformation inside the derived category of a Deligne-Mumford stack). Let $\mathcal{T} \subset D^b(X)$ be a full admissible subcategory. Given a smooth algebraic variety B , one wants to define the deformation of \mathcal{T} inside $D^b(X)$ over B .

Definition 3.3.1 *Let X be a smooth projective variety, let $\mathcal{T} \subset D^b(X)$ be a full admissible subcategory and B a smooth algebraic variety with a marked point $0 \in B$. A **smooth deformation of \mathcal{T} inside X over B** is the data of:*

- a smooth projective morphism $\pi : \mathcal{X} \rightarrow B$ such that $\mathcal{X}_0 = X$,
- a full admissible subcategory $\mathcal{D} \subset D^b(\mathcal{X})$, which is B -linear, such that $\mathcal{E}_0 := \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X} \times_B \mathcal{X}}} \mathcal{O}_{\mathcal{X}_0 \times \mathcal{X}_0} \in D^b(\mathcal{X}_0 \times \mathcal{X}_0)$ is the kernel of the projection $D^b(\mathcal{X}_0) \rightarrow \mathcal{T}$, where $\mathcal{E} \in D^b(\mathcal{X} \times_B \mathcal{X})$ is the kernel representing the projection functor $D^b(\mathcal{X}) \rightarrow \mathcal{D}$.

The existence of the kernels in the above definition has been proved by Kuznetsov in [Kuz11]. We have a semi-orthogonal decomposition $D^b(\mathcal{X}) = \langle \mathcal{D}, {}^\perp \mathcal{D} \rangle$ and I denote by ${}^\perp \mathcal{E} \in D^b(\mathcal{X} \times_B \mathcal{X})$ the kernel of the projection $D^b(\mathcal{X}) \rightarrow {}^\perp \mathcal{D}$. Let us display a Cartesian diagram which will be important to study the deformation of \mathcal{T} over B .

$$\begin{array}{ccccc}
 & & \mathcal{X} \times_B \mathcal{X} & & \\
 & \swarrow p & \uparrow j_b & \searrow q & \\
 \mathcal{X} & & \mathcal{X}_b \times \mathcal{X}_b & & \mathcal{X} \\
 \uparrow i_b & \swarrow p_b & & \searrow q_b & \uparrow i_b \\
 \mathcal{X}_b & & & & \mathcal{X}_b
 \end{array}$$

Proposition 3.3.2 *With hypotheses and notation as above, for all $b \in B$, there exists a semi-orthogonal decomposition:*

$$\mathrm{D}^b(\mathcal{X}_b) = \langle \mathcal{D}_b, {}^t\mathcal{D}_b \rangle,$$

where \mathcal{D}_b (resp. ${}^t\mathcal{D}_b$) is the full subcategory of $\mathrm{D}^b(\mathcal{X}_b)$ closed under taking direct summands which is generated by the objects $\mathbf{R}p_{b*}(\mathbf{L}q_b^*\mathcal{F} \otimes \mathcal{E}_b)$ (resp. $\mathbf{R}p_{b*}(\mathbf{L}q_b^*\mathcal{F} \otimes {}^\perp\mathcal{E}_b)$), for $\mathcal{F} \in \mathrm{D}^b(X_b)$.

This proposition allows one to think of the \mathcal{D}_b for $b \in B$ as the deformation of $\mathcal{D}_0 = \mathcal{T}$ over B .

Proof:

► Since $\mathcal{X} \rightarrow B$ is projective, the family of line bundles $\mathcal{O}_{\mathcal{X}/B}(m)|_{\mathcal{X}_b}, m \in \mathbb{N}$ generates $\mathrm{D}^b(\mathcal{X}_b)$. Hence, the category \mathcal{D}_b (resp. ${}^t\mathcal{D}_b$) is also generated by the $\mathbf{R}p_{b*}(\mathbf{L}q_b^*\mathbf{L}i_b^*\mathcal{F} \otimes \mathcal{E}_b)$ (resp. $\mathbf{R}p_{b*}(\mathbf{L}q_b^*\mathbf{L}i_b^*\mathcal{F} \otimes {}^\perp\mathcal{E}_b)$), for $\mathcal{F} \in \mathrm{D}^b(\mathcal{X})$. I first prove that ${}^t\mathcal{D}_b$ is left orthogonal to \mathcal{D}_b . Let $\mathcal{F}, \mathcal{G} \in \mathrm{D}^b(\mathcal{X})$, we have:

$$\begin{aligned} & \mathrm{Hom}(\mathbf{R}p_{b*}(\mathbf{L}q_b^*(\mathbf{L}i_b^*\mathcal{F}) \otimes {}^\perp\mathcal{E}_b), \mathbf{R}p_{b*}(\mathbf{L}q_b^*(\mathbf{L}i_b^*\mathcal{G}) \otimes \mathcal{E}_b)) \\ &= \mathrm{Hom}(\mathbf{R}p_{b*}(\mathbf{L}j_b^*(\mathbf{L}q^*\mathcal{F} \otimes {}^\perp\mathcal{E})), \mathbf{R}p_{b*}(\mathbf{L}j_b^*(\mathbf{L}q^*\mathcal{G} \otimes \mathcal{E}))) \\ &= \mathrm{Hom}(\mathbf{L}i_b^*(\mathbf{R}p_*(\mathbf{L}q^*\mathcal{F} \otimes {}^\perp\mathcal{E})), \mathbf{L}i_b^*(\mathbf{R}p_*(\mathbf{L}q^*\mathcal{G} \otimes \mathcal{E}))) \\ &= \mathrm{Hom}(\mathbf{R}p_*(\mathbf{L}q^*\mathcal{F} \otimes {}^\perp\mathcal{E}), \mathbf{R}i_{b*}(\mathbf{L}i_b^*(\mathbf{R}p_*(\mathbf{L}q^*\mathcal{G} \otimes \mathcal{E})))) \\ &= \mathrm{Hom}(\mathbf{R}p_*(\mathbf{L}q^*\mathcal{F} \otimes {}^\perp\mathcal{E}), \mathbf{R}p_*(\mathbf{L}q^*\mathcal{G} \otimes \mathcal{E}) \otimes \mathbf{R}i_{b*}\mathcal{O}_{\mathcal{X}_b}), \end{aligned}$$

here the first equality is the identity $\mathbf{L}q_b^*\mathbf{L}i_b^* = \mathbf{L}j_b^*\mathbf{L}q^*$, the second is the flat base change $\mathbf{R}p_{b*}\mathbf{L}j_b^* = \mathbf{L}i_b^*\mathbf{R}p_*$, the third is adjunction with respect to i_b and the fourth is the projection formula with respect to i_b . By flat base change for the morphism $\pi : \mathcal{X} \rightarrow B$, we have $\mathbf{R}i_{b*}\mathcal{O}_{\mathcal{X}_b} = \mathbf{L}\pi^*\mathbb{C}(b)$. The category \mathcal{D} is B -linear by hypothesis, so that $\mathbf{R}p_*(\mathbf{L}q^*\mathcal{G} \otimes \mathcal{E}) \otimes \mathbf{R}i_{b*}\mathcal{O}_{\mathcal{X}_b} \in \mathcal{D}$. As a consequence, we deduce the vanishing:

$$\mathrm{Hom}(\mathbf{R}p_*(\mathbf{L}q^*\mathcal{F} \otimes {}^\perp\mathcal{E}), \mathbf{R}p_*(\mathbf{L}q^*\mathcal{G} \otimes \mathcal{E}) \otimes \mathbf{R}i_{b*}\mathcal{O}_{\mathcal{X}_b}) = 0.$$

As \mathcal{D}_b (resp. ${}^t\mathcal{D}_b$) is the full subcategory of $\mathrm{D}^b(\mathcal{X}_b)$ closed under taking direct summands which is generated by the $\mathbf{R}p_{b*}(\mathbf{L}q_b^*\mathbf{L}i_b^*\mathcal{F} \otimes \mathcal{E}_b)$ (resp. $\mathbf{R}p_{b*}(\mathbf{L}q_b^*\mathbf{L}i_b^*\mathcal{F} \otimes {}^\perp\mathcal{E}_b)$) for $\mathcal{F} \in \mathrm{D}^b(\mathcal{X})$, the above vanishing finally proves that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$, for all $\mathcal{G} \in \mathcal{D}_b$ and $\mathcal{F} \in {}^t\mathcal{D}_b$.

We are left to show that for all $\mathcal{H} \in \mathrm{D}^b(X_b)$, there exists an exact triangle:

$$\mathcal{G} \rightarrow \mathcal{H} \rightarrow \mathcal{F},$$

with $\mathcal{F} \in {}^t\mathcal{D}_b$ and $\mathcal{G} \in \mathcal{D}_b$. But on $\mathcal{X} \times_B \mathcal{X}$, we have an exact triangle:

$$\mathcal{E} \rightarrow \mathcal{O}_{\Delta/B} \rightarrow {}^\perp\mathcal{E}.$$

Hence, for all $\mathcal{F} \in \mathrm{D}^b(X_b)$, we have an exact triangle:

$$\mathbf{R}p_{b*}\mathbf{L}(q_b^*(\mathcal{F}) \otimes \mathcal{E}_b) \rightarrow \mathcal{F} \rightarrow \mathbf{R}p_{b*}\mathbf{L}(q_b^*(\mathcal{F}) \otimes {}^\perp\mathcal{E}_b).$$

◀

Corollary 3.3.3 *For all $b \in B$, the objects \mathcal{E}_b (resp. ${}^\perp \mathcal{E}_b$) is the kernel of the projection functor $D^b(\mathcal{X}_b) \rightarrow \mathcal{D}_b$ (resp. $D^b(\mathcal{X}_b) \rightarrow {}^\perp \mathcal{D}_b$).*

Proof :

► Using exactly the same identities as in the proof of proposition 3.3.2, one shows that $\mathbf{R}p_{b*}(\mathbf{L}q_b^*(\mathcal{G}) \otimes \mathcal{E}_b)$ is quasi-isomorphic to \mathcal{G} if $\mathcal{G} \in \mathcal{D}_b$ and is zero if $\mathcal{G} \in {}^t \mathcal{D}_b$ (the opposite holds for ${}^\perp \mathcal{E}_b$). Since Proposition 3.3.2 shows that ${}^t \mathcal{D}_b = {}^\perp \mathcal{D}_b$, the claim is proved. ◀

Theorem 3.3.4 *Let X be a smooth projective variety and $\mathcal{T} \subset D^b(X)$ a full admissible subcategory. Let B a smooth variety and \mathcal{D} be a smooth deformation of \mathcal{T} over B . The dimension of the Hochschild homology of \mathcal{D}_b is constant for $b \in B$.*

Note that we do not need to assume that the kernel of the projection $D^b(\mathcal{X}) \rightarrow \mathcal{D}$ is flat over B .

Proof :

► Let $\mathcal{X} \rightarrow B$ be a smooth projective morphism such that \mathcal{D} is a full admissible subcategory of $D^b(\mathcal{X})$. Let $\mathcal{E} \in D^b(\mathcal{X} \times_B \mathcal{X})$ be the kernel of the projection functor $D^b(\mathcal{X}) \rightarrow \mathcal{D}$. By corollary 3.3.3, we know that for all $b \in B$, the object $\mathcal{E}_b \in D^b(\mathcal{X}_b \times \mathcal{X}_b)$ is the kernel of the projection functor $D^b(\mathcal{X}_b) \rightarrow \mathcal{D}_b$. As a consequence of Theorem 4.5 in [Kuz09], we have an equality:

$$\mathrm{HH}_\bullet(\mathcal{D}_b) = H^\bullet(\mathcal{X}_b \times \mathcal{X}_b, \mathcal{E}_b \otimes \mathcal{E}_b^T),$$

where \mathcal{E}_b^T is the pull back of \mathcal{E}_b with respect to the permutation $\mathcal{X}_b \times \mathcal{X}_b \rightarrow \mathcal{X}_b \times \mathcal{X}_b$. Let us prove that the dimension of the cohomology vector spaces $H^i(\mathcal{X}_b \times \mathcal{X}_b, \mathcal{E}_b \otimes \mathcal{E}_b^T)$ are upper semi-continuous with respect to $b \in B$ for all i . By flat base change for the diagram:

$$\begin{array}{ccc} \mathcal{X}_b \times \mathcal{X}_b & \xrightarrow{j_b} & \mathcal{X} \times_B \mathcal{X} \\ \downarrow \pi_b & & \downarrow \pi \\ \mathrm{Spec}(\mathbb{C}(b)) & \longrightarrow & B \end{array}$$

we have the equality:

$$H^i(\mathcal{X}_b \times \mathcal{X}_b, \mathcal{E}_b \otimes \mathcal{E}_b^T) = \mathcal{H}^i(\mathbf{R}\pi_*(\mathcal{E} \otimes \mathcal{E}^T) \otimes \mathbb{C}(b)).$$

Since B is a smooth variety, we can represent $\mathbf{R}\pi_*(\mathcal{E} \otimes \mathcal{E}^T) \otimes \mathbb{C}(b)$ by a bounded complex of vector bundles on B , say E^\bullet . Thus, we only have to show the following : the cohomology sheaves of $E^\bullet \otimes \mathbb{C}(b)$ are upper semi-continuous, for $b \in B$. This result is now obvious as the dimension of the image of the differential:

$$d^\bullet \otimes \mathbb{C}(b) : E^\bullet \otimes \mathbb{C}(b) \rightarrow E^{\bullet+1} \otimes \mathbb{C}(b)$$

is lower semi-continuous with respect to B .

We have proved that the dimension of $\mathrm{HH}_i(\mathcal{D}_b)$ is upper semi-continuous with respect to B , for all i . This holds also true for the dimension $\mathrm{HH}_i({}^\perp \mathcal{D}_b)$. By corollary 7.5 of [Kuz09], we have:

$$\mathrm{HH}_i(\mathcal{D}_b) \oplus \mathrm{HH}_i({}^\perp \mathcal{D}_b) = \mathrm{HH}_i(\mathrm{D}^b(\mathcal{X}_b)).$$

But the morphism $\mathcal{X} \rightarrow B$ is smooth projective, so that the Hodge numbers of \mathcal{X}_b are constant with respect to B . By the Hochschild-Kostant-Rosenberg decomposition, this implies that the Hochschild numbers of \mathcal{X}_b are constant. Hence the sum of the dimensions of the cohomology vector spaces $\mathrm{HH}^i(\mathcal{D}_b)$ and $\mathrm{HH}^i({}^\perp \mathcal{D}_b)$ is constant with respect to B . But each dimension is upper semi-continuous with respect to B , so that they are in fact both constant with respect to B . ◀

Before going turning to deformation results for compact hyper-Kähler categories, I want to comment about the level of generality of the deformation theory used above. In order to define the notion of deformation of an admissible subcategory $\mathcal{T} \subset \mathrm{D}^b(X)$, one could be tempted to work with a seemingly more general definition, as follows. A deformation of \mathcal{T} over B is the data of a morphism $\pi : \mathcal{X} \rightarrow B$ and a B -linear admissible subcategory $\mathcal{D} \subset \mathrm{D}^b(\mathcal{X} \times_B \mathcal{X})$, such that $\mathcal{D}_0 = \mathcal{T}$ and the flat base change formula holds for \mathcal{D} with respect to the diagram:

$$\begin{array}{ccc} \mathcal{D}_b & \xrightarrow{\mathbf{R}j_{b*}} & \mathcal{D} \\ \uparrow \mathbf{L}\tilde{\pi}_b^* & & \uparrow \mathbf{L}\tilde{\pi}^* \\ \mathrm{Spec}(\mathbb{C}(b)) & \xrightarrow{k_{b*}} & B \end{array}$$

The base change formula would imply $\mathbf{R}j_{b*}\mathbf{L}\tilde{\pi}_b^*\mathbb{C}(b) = \mathbf{L}\tilde{\pi}^*\mathbf{R}k_{b*}\mathbb{C}(b)$. But we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{D}_d & \xrightarrow{\quad} & \mathrm{D}^b(\mathcal{X}_b) \\ & \searrow \mathbf{R}\tilde{\pi}_{b*} & \downarrow \\ & & \mathrm{D}^b(\mathrm{Spec}(\mathbb{C}(b))) \end{array}$$

Assume that $\mathcal{O}_{\mathcal{X}} \subset \mathcal{D}$. The fact that \mathcal{D} is B -linear then implies $\mathbf{L}\tilde{\pi}_b^*\mathbb{C}(b) = \mathcal{O}_{X_b}$. As a consequence, we have $\mathbf{L}\tilde{\pi}^*\mathbf{R}k_{b*}\mathbb{C}(b) = j_{b*}\mathcal{O}_{X_b}$. In particular, we have $\mathrm{Tor}_B^1(\mathcal{O}_{\mathcal{X}}, \mathbb{C}(b)) = 0$. By Theorem 22.3 of [Mat89], the morphism $\mathcal{X} \rightarrow B$ is flat. Hence, a more general setting than the one developed above for the deformation of triangulated categories can not be obtained if one requires the following three conditions:

- the total space of the deformation is a full admissible subcategory of the derived category of an algebraic variety,
- the base change formula holds for the total space of the deformation,
- $\mathcal{O}_{\mathcal{X}} \in \mathcal{D}$.

The first two conditions seem essential if one wants to get some significant homological results while working with admissible subcategories of derived categories of algebraic

varieties. As far as the third condition is concerned, it is satisfied in many examples (for instance in the setting of non-commutative resolution of singularities).

I will now focus on the deformation theory of compact hyper-Kähler categories. We start with the following:

Proposition 3.3.5 *Let X be a smooth projective variety, let $\mathcal{T} \subset \mathrm{D}^b(X)$ be a full admissible subcategory and B a smooth algebraic variety. Let \mathcal{D} be a smooth deformation of \mathcal{T} over B . Assume that $\mathcal{O}_{\mathcal{X}} \in \mathcal{D}$ and that \mathcal{T} is a compact hyper-Kähler category. Then, there exists a neighborhood $0 \in U \subset B$, such that \mathcal{D}_b is compact hyper-Kähler for all $b \in U$.*

Proof :

► Let $\pi : \mathcal{X} \rightarrow B$ be the smooth projective morphism in which the deformation \mathcal{D} is embedded. We know that $\mathcal{D}_0 = \mathcal{T}$ is compact hyper-Kähler (of dimension $2m$). In particular, the category \mathcal{T} is Calabi-Yau of dimension $2m$. Hence there exists a quasi-isomorphism:

$$\theta_0 : \mathcal{E}_0 \rightarrow \mathcal{E}_0 \otimes q_0^* \omega_{\mathcal{X}_0} [\dim X_0 - 2m].$$

But $\mathcal{E}_0 = \mathcal{E} \otimes_{\mathcal{X} \times_B \mathcal{X}} \mathcal{O}_{\mathcal{X}_0 \times \mathcal{X}_0}$ and $\omega_{\mathcal{X}_0} = \omega_{\mathcal{X}/B} \otimes_{\mathcal{X} \times_B \mathcal{X}} \mathcal{O}_{\mathcal{X}_0 \times \mathcal{X}_0}$. Hence, by Nakayama's lemma, there exists a neighborhood $0 \in U \subset B$, such that θ_0 can be lifted to a quasi-isomorphism:

$$\theta_U : \mathcal{E} \otimes_{\mathcal{X} \times_B \mathcal{X}} \mathcal{O}_U \rightarrow \mathcal{E} \otimes_{\mathcal{X} \times_B \mathcal{X}} q^* \omega_{\mathcal{X}_U/U} [\dim(\mathcal{X}_U/U) - 2m].$$

This proves that the categories $\mathcal{D}_b, b \in U$ are Calabi-Yau of dimension $2m$. Since X_b is smooth projective for all $b \in B$, the categories \mathcal{D}_b are also smooth, compact and regular for all $b \in B$. It remains to prove (up to shrinking U), that $\mathbb{C}[t]/(t^{m+1})$ (with t homogeneous of degree 2) is a homological unit for $\mathcal{D}_b, b \in U$. Since \mathcal{D} contains $\mathcal{O}_{\mathcal{X}}$ and is B -linear, the categories $\mathcal{D}_b, b \in B$ all contain $\mathcal{O}_{\mathcal{X}_b}$. We deduce that for all $b \in B$, the graded algebra $H^\bullet(\mathcal{O}_{\mathcal{X}_b})$ is a homological unit for \mathcal{D}_b . But $H^\bullet(\mathcal{O}_{X_0}) \simeq \mathbb{C}[t]/(t^{m+1})$. Hence there exists another neighborhood $0 \in U' \subset B$ such that $H^\bullet(\mathcal{O}_{X_b}) \simeq \mathbb{C}[t]/(t^{m+1})$, for all $b \in U'$. Taking $U'' = U \cap U'$ gives the neighborhood we are looking for. ◀

The above statement shows that being compact hyper-Kähler is an open condition (if one assumes that $\mathcal{O}_{\mathcal{X}} \in \mathcal{D}$). I also expect it to be a closed condition. Namely:

Conjecture 3.3.6 *Let X be a smooth projective variety, let $\mathcal{T} \subset \mathrm{D}^b(X)$ be a full admissible subcategory and B a smooth algebraic variety. Let \mathcal{D} be a deformation of \mathcal{T} over B . Assume that \mathcal{D}_b is compact hyper-Kähler for all $b \neq 0$. Then, the category $\mathcal{D}_0 = \mathcal{T}$ is compact hyper-Kähler.*

The commutative specialization of this result is well-known. Namely, let $\pi : \mathcal{X} \rightarrow B$ be a smooth projective morphism with B smooth. If \mathcal{X}_b is compact hyper-Kähler for all $b \neq 0$, then \mathcal{X}_0 is also compact hyper-Kähler. It is usually proved using the holonomy principle and the invariance of holonomy groups in smooth families (see [Huy99], section 1). As far as I am aware, there are no algebraic proof of this result. Hence, a proof of conjecture 3.3.6, would certainly require the design of interesting new categorical techniques.

Two key results are to be proved in order to demonstrate conjecture 3.3.6 : the invariance of the Calabi-Yau condition and of the homological unit under smooth deformations.

Conjecture 3.3.7 *Let X be a smooth projective variety, let $\mathcal{T} \subset \mathrm{D}^b(X)$ be a full admissible subcategory and B a smooth algebraic variety. Let \mathcal{D} be a deformation of \mathcal{T} over B . Assume that \mathcal{D}_b is Calabi-Yau of dimension r for all $b \neq 0$. Then, the category $\mathcal{D}_0 = \mathcal{T}$ is Calabi-Yau of dimension r .*

Note that it is very unlikely that this conjecture can be proved by abstract algebraic arguments. Indeed, the work of Keller ([Kel11]) suggests that strong additional hypotheses are usually used in order to prove that a deformation of a Calabi-Yau algebra is again Calabi-Yau. Hence, the fact that the categories appearing in conjecture 3.3.7 are subcategories of derived categories of algebraic varieties will certainly play an important role in a potential proof.

I will provide a sketch of a proof of conjecture 3.3.7 if some additional hypotheses are satisfied. First of all, the \mathcal{D}_b are Calabi-Yau of dimension r for all $b \neq 0$, hence there exists a quasi-isomorphism:

$$\psi_b : \mathcal{E}_b \otimes q_b^* \omega_{\mathcal{X}_b}[\dim \mathcal{X}_b - r] \rightarrow \mathcal{E}_b,$$

for all $b \neq 0$. Mimicking the proof of Theorem 3.3.4, one can prove that dimension of the cohomology vector spaces $\mathrm{Hom}(\mathcal{E}_b \otimes q_b^* \omega_{\mathcal{X}_b}[\dim \mathcal{X}_b - r], \mathcal{E}_b)$ is upper semi-continuous with respect to B . Hence, we have a non zero morphism:

$$\psi_0 : \mathcal{E}_0 \otimes q_0^* \omega_{\mathcal{X}_0}[\dim \mathcal{X}_0 - r] \rightarrow \mathcal{E}_0.$$

One would like to show that this morphism gives a non-zero natural transformation of functors $S_{\mathcal{D}_0}[-r] \rightarrow \mathrm{id}_{\mathcal{D}_0}$, where $S_{\mathcal{D}_0}$ is the Serre functor of \mathcal{D}_0 . Unfortunately, it is known that non-zero morphisms between kernels might give vanishing natural transformations between their associated Fourier-Mukai functors.⁵ But even knowing that there exists a non-zero natural transformation $S_{\mathcal{D}_0}[-r] \rightarrow \mathrm{id}_{\mathcal{D}_0}$ would not be sufficient for the proof I envision. I would also need that for any generating family of objects in $(\mathcal{U}_i)_{i \in I} \in \mathcal{D}_0$, there exists $i \in I$ such that

$$\psi_0(\mathcal{U}_i) : S_{\mathcal{D}_0}[-r](\mathcal{U}_i) \rightarrow \mathcal{U}_i$$

is non zero. This condition obviously implies that the natural transformation associated to ψ_0 is non-zero but it is not equivalent to it. Indeed, in a triangulated category, one can find commutative diagrams of exact triangles:

$$\begin{array}{ccccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & A_1[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & A_2[1] \end{array}$$

with f and g being zero and h not zero. Hence, the fact that a natural transformation vanishes on a generating set of objects of \mathcal{D}_b does not necessarily implies that it vanishes on all objects of \mathcal{D}_b . Finally, even if a morphism $f : \mathcal{A}^\bullet \rightarrow A_1^\bullet$ is non-zero, one can not

⁵Let E be an elliptic curve. There exists a non vanishing morphism $\Delta \rightarrow \Delta[2]$ on $E \times E$. But it is easy to prove that all natural transformations $\mathrm{id}_{\mathrm{D}^b(E)} \rightarrow \mathrm{id}_{\mathrm{D}^b(E)}[2]$ vanish.

deduce that there exists $i \in \mathbb{Z}$, such that $\mathcal{H}^i(f) \neq 0$. One can only deduce that f is homotopically equivalent to 0. But we know that for all $b \neq B$ and all complexes $\mathcal{F} \in \mathcal{D}_b$ such that $\mathcal{H}^0(\mathcal{F}) \neq 0$, the map:

$$\mathcal{H}^0(\psi_b) : \mathcal{H}^0(S_{\mathcal{D}_b}[-r](\mathcal{F})) \rightarrow \mathcal{H}^0(\mathcal{F}),$$

is non-zero : this is this kind of non-vanishing one would like to get at the limit.

Hypothesis 1 : *Assume that the categories $\mathcal{D}_b, b \neq 0$ are Calabi-Yau of dimension r . Then, there exists $\psi_0 \in \text{Hom}(\mathcal{E}_0 \otimes q_b^* \omega_{\mathcal{X}_0}[\dim \mathcal{X}_0 - r], \mathcal{E}_0)$ with the property that for all generating family of objects $(\mathcal{U}_i)_{i \in I} \in \mathcal{D}_0$, there exists $i_1 \in I$ such that for some $j \in \mathbb{Z}$, the induced map:*

$$\mathcal{H}^j(\psi_0)(\mathcal{U}_{i_1}) : \mathcal{H}^j(S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_1})) \rightarrow \mathcal{H}^j(\mathcal{U}_{i_1})$$

is non zero.

It seems to me that the existence of $\psi_0 \in \text{Hom}(\mathcal{E}_0 \otimes q_0^* \omega_{\mathcal{X}_0}[\dim \mathcal{X}_0 - r], \mathcal{E}_0)$ such that the induced natural transformation $S_{\mathcal{D}_0}[-r] \rightarrow \text{id}_{\mathcal{D}_0}$ is non-zero could be best proved in the context of DG-categories (see [Toë07]).

Hypothesis 2: *There exists a generating family of objects, say $(\mathcal{U}_i)_{i \in I} \in \mathcal{D}_0$, which are concentrated in degree 0 and torsion free, satisfying the following conditions:*

- $\text{Hom}(\mathcal{U}_i, \mathcal{U}_i) = \mathbb{C}$ (the \mathcal{U}_i are simple objects),
- *there exists $i_0 \in I$ such that for all $i \in I$, there is a non-zero morphism $\mathcal{U}_{i_0} \rightarrow \mathcal{U}_i$.*

Note that hypothesis 1 is **necessary** for conjecture 3.3.7 to be true. Indeed, if this conjecture is true, there exists $\phi_0 \in \text{HH}_{-r}(\mathcal{D}_0)$ such that the associated natural transformation is an isomorphism. Hence, hypothesis 1 is satisfied for ϕ_0 . Hypothesis 2 is probably superfluous and I don't believe it is of crucial importance for conjecture 3.3.7 to be true. It happens that it is however a very convenient assumption to prove conjecture 3.3.7. Furthermore this condition is satisfied in a number of interesting geometric cases that are described below.

Let $\mathcal{D}_0 = \text{D}^b(X)$, where X is a smooth projective variety. For k big enough (say $k \geq k_0$), the spaces of sections $H^0(\mathcal{O}_X(k))$ are non trivial. Hence the family $(\mathcal{O}_X(mk_0))_{m \in \mathbb{N}}$ satisfies hypothesis 2.

Another example comes the theory of non-commutative variety. Assume there exists a morphism $f : \mathcal{X}_0 \rightarrow Y_0$, with Y_0 projective (but not necessarily smooth) and exceptional bundles E_1, \dots, E_q with respect to f on \mathcal{X}_0 such that:

$$\mathcal{D}_0 \simeq \text{D}^{\text{perf}}(Y_0, f_* \mathcal{E}nd(E_1 \oplus \dots \oplus E_q)).$$

Then, the family $(f^* \mathcal{O}_Y(mk_0))_{m \in \mathbb{N}} \cup \{E_1(k_0), \dots, E_q(k_0)\}$ is a generating family of simple objects for \mathcal{D}_0 , for any $k_0 \geq 1$. Furthermore, for k_0 big enough, the spaces of sections $H^0(f^* \mathcal{O}_Y(mk_0))$ and $H^0(E_j(k_0))$ are non-trivial, for all $m \in \mathbb{N}$ and all $1 \leq j \leq q$. Hence the family $(f^* \mathcal{O}_Y(mk_0))_{m \in \mathbb{N}} \cup \{E_1(k_0), \dots, E_q(k_0)\}$ satisfies hypothesis 2. This is typically the situation one encounters when \mathcal{D}_0 is a non-commutative resolution of singularities of Y_0 (see [Kuz08b, BLVdB10]).

Proposition 3.3.8 *Let X be a smooth projective variety, let $\mathcal{T} \subset \mathrm{D}^b(X)$ be a full admissible subcategory and B a smooth algebraic variety. Let \mathcal{D} be a smooth deformation of \mathcal{T} over B . Assume that \mathcal{D}_b is Calabi-Yau of dimension $r = \dim X$ for all $b \neq 0$ and that hypothesis 1 and 2 above are satisfied. Then, the category $\mathcal{D}_0 = \mathcal{T}$ is Calabi-Yau of dimension r .*

Proof:

► Let $\pi : \mathcal{X} \rightarrow B$ be the smooth morphism associated to the deformation \mathcal{D} , let $b \in B$ and let $\delta_b : \mathcal{D}_b \rightarrow \mathrm{D}^b(\mathcal{X}_b)$ be the embedding functor. Let $s \in H^0(\mathcal{X}_b, \omega_{\mathcal{X}_b})$. For all $\mathcal{F} \in \mathcal{D}_b$, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Phi(\mathcal{F})} & \delta_b \delta_b^! (\mathcal{F} \otimes \omega_{\mathcal{X}_b}) \\ & \searrow id \otimes s & \downarrow adj \\ & & \mathcal{F} \otimes \omega_{\mathcal{X}_b} \end{array}$$

where adj is the adjunction morphism and Φ is the natural transformation of functors $id \rightarrow S_{\mathcal{D}_b}[-r]$ associated to $\gamma(s)$ where γ is the projection $\gamma : H^0(\mathcal{X}_b, \omega_{\mathcal{X}_b}) \simeq \mathrm{HH}_{-r}(\mathrm{D}^b(\mathcal{X}_b)) \rightarrow \mathrm{HH}_{-r}(\mathcal{D}_b)$. Note that $\delta_b \delta_b^! (\mathcal{F} \otimes \omega_{\mathcal{X}_b}) = S_{\mathcal{D}_b}[-r](\mathcal{F})$. We know that for all $b \neq 0$, there exists $\Phi_b \in \mathrm{HH}_{-r}(\mathcal{D}_b)$ such that the associated natural transformation $\Phi_b : id_{\mathcal{D}_b} \rightarrow S_{\mathcal{D}_b}[-r]$ is an isomorphism. Hence $H^0(\mathcal{X}_b, \omega_{\mathcal{X}_b}) \neq 0$, for all $b \neq 0$. As a consequence, we have $H^0(\mathcal{X}_0, \omega_{\mathcal{X}_0}) \neq 0$ and we fix s_0 a non-vanishing section of $\omega_{\mathcal{X}_0}$.

By hypothesis 2, there exists a generating family $(\mathcal{U}_i)_{i \in I} \in \mathcal{D}_0$ of simple objects concentrated in degree 0 and torsion free with $i_0 \in I$ such that for all $i \in I$, there is a non-zero morphism $t_0 : \mathcal{U}_{i_0} \rightarrow \mathcal{U}_i$. By hypothesis 1, there exists $\Psi \in \mathrm{Hom}(S_{\mathcal{D}_0}[-r], id_{\mathcal{D}_0})$ and $i_1 \in I$ such that the morphism:

$$\mathcal{H}^0(\Psi(\mathcal{U}_{i_1})) : \mathcal{H}^0(S_{\mathcal{D}_0}(\mathcal{U}_{i_1})[-r]) \rightarrow \mathcal{H}^0(\mathcal{U}_{i_1})$$

is non-zero. Consider the commutative diagram:

$$\begin{array}{ccccc} S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_1}) & \xrightarrow{\Psi(\mathcal{U}_{i_1})} & \mathcal{U}_{i_1} & \xrightarrow{\Phi(\mathcal{U}_{i_1})} & S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_1}) \\ & & \searrow id \otimes s_0 & & \downarrow adj \\ & & & & S_{\mathrm{D}^b(\mathcal{X}_0)}[-r](\mathcal{U}_{i_1}) \end{array}$$

Let us prove that the composition $id \otimes s_0 \circ \Psi(\mathcal{U}_{i_1})$ is non zero. Since \mathcal{U}_{i_1} is concentrated in degree 0 and torsion free, the map $id \otimes s_0 : \mathcal{U}_{i_1} \rightarrow \mathcal{U}_{i_1} \otimes \omega_0$ is injective. But the map $\mathcal{H}^0(\Psi(\mathcal{U}_{i_1})) : \mathcal{H}^0(S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_1})) \rightarrow \mathcal{H}^0(\mathcal{U}_{i_1})$ is non-zero. This demonstrates that $id \otimes s_0 \circ \Psi(\mathcal{U}_{i_1})$ is non zero. It follows that the composition:

$$\Phi(\mathcal{U}_{i_1}) \circ \Psi(\mathcal{U}_{i_1}) : S_{D_0}[-r](\mathcal{U}_{i_1}) \rightarrow S_{D_0}[-r](\mathcal{U}_{i_1})$$

is non-zero. But we have $\text{Hom}(S_{D_0}[-r](\mathcal{U}_{i_1}), S_{D_0}[-r](\mathcal{U}_{i_1})) = \text{Hom}(\mathcal{U}_{i_1}, \mathcal{U}_{i_1}) = \mathbb{C}.id$. As a consequence, the composition $\Phi(\mathcal{U}_{i_1}) \circ \Psi(\mathcal{U}_{i_1})$ is a non-vanishing multiple of the identity. Since $\Phi(\mathcal{U}_{i_1})$ induces an injective map in cohomology (because it factors $id \otimes s$), we deduce that $\Phi(\mathcal{U}_{i_1})$ is a quasi-isomorphism. And thus $\Phi(\mathcal{U}_{i_1})$ is also a quasi-isomorphism, which is inverse to $\Psi(\mathcal{U}_{i_1})$ (up to a non-zero scalar).

Let us prove that the same holds for $\Phi(\mathcal{U}_{i_0})$ and $\Psi(\mathcal{U}_{i_0})$. We have a commutative diagram:

$$\begin{array}{ccccccc} S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_0}) & \xrightarrow{\Psi(\mathcal{U}_{i_0})} & \mathcal{U}_{i_0} & \xrightarrow{\Phi(\mathcal{U}_{i_0})} & S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_0}) & \xrightarrow{\Psi(\mathcal{U}_{i_0})} & \mathcal{U}_{i_0} \\ \downarrow S_{\mathcal{D}_0}[-r](t_{i_1}) & & \downarrow t_{i_1} & & \downarrow S_{\mathcal{D}_0}[-r](t_{i_1}) & & \downarrow t_{i_1} \\ S_{\mathcal{D}_0}(\mathcal{U}_{i_1})[-r] & \xrightarrow{\Psi(\mathcal{U}_{i_1})} & \mathcal{U}_{i_1} & \xrightarrow{\Phi(\mathcal{U}_{i_1})} & S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_1}) & \xrightarrow{\Psi(\mathcal{U}_{i_1})} & \mathcal{U}_{i_1} \end{array}$$

Since the composite $\Phi(\mathcal{U}_{i_1}) \circ \Psi(\mathcal{U}_{i_1})$ and $\Psi(\mathcal{U}_{i_1}) \circ \Phi(\mathcal{U}_{i_1})$ are isomorphism and the maps t_{i_1} and $S_{\mathcal{D}_0}[-r](t_{i_1})$ are non-zero, we find that both compositions $\Phi(\mathcal{U}_{i_0}) \circ \Psi(\mathcal{U}_{i_0})$ and $\Psi(\mathcal{U}_{i_0}) \circ \Phi(\mathcal{U}_{i_0})$ are non-zero. But $\text{Hom}(S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_0}), S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_0})) = \text{Hom}(\mathcal{U}_{i_0}, \mathcal{U}_{i_0}) = \mathbb{C}.id$. As a consequence, the composition $\Phi(\mathcal{U}_{i_0}) \circ \Psi(\mathcal{U}_{i_0})$ and $\Psi(\mathcal{U}_{i_0}) \circ \Phi(\mathcal{U}_{i_0})$ are non-zero multiple of the identity. This proves that $\Phi(\mathcal{U}_{i_0})$ and $\Psi(\mathcal{U}_{i_0})$ are both isomorphisms, which are (up to a non-zero scalar) inverse to each other.

We conclude by proving that for all $i \in I$, the maps $\Psi(\mathcal{U}_i)$ and $\Phi(\mathcal{U}_i)$ are isomorphisms. Indeed, for all $i \in I$, we have a commutative diagram:

$$\begin{array}{ccccccc} S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_0}) & \xrightarrow{\Psi(\mathcal{U}_{i_0})} & \mathcal{U}_{i_0} & \xrightarrow{\Phi(\mathcal{U}_{i_0})} & S_{\mathcal{D}_0}[-r](\mathcal{U}_{i_0}) & \xrightarrow{\Psi(\mathcal{U}_{i_0})} & \mathcal{U}_{i_0} \\ \downarrow S_{\mathcal{D}_0}[-r](t_i) & & \downarrow t_i & & \downarrow S_{\mathcal{D}_0}[-r](t_i) & & \downarrow t_i \\ S_{\mathcal{D}_0}(\mathcal{U}_i)[-r] & \xrightarrow{\Psi(\mathcal{U}_i)} & \mathcal{U}_i & \xrightarrow{\Phi(\mathcal{U}_i)} & S_{\mathcal{D}_0}[-r](\mathcal{U}_i) & \xrightarrow{\Psi(\mathcal{U}_i)} & \mathcal{U}_i \end{array}$$

with t_i and $S_{\mathcal{D}_0}(t_i)$ non zero. Using the same arguments as before, we show that $\Phi(\mathcal{U}_i)$ and $\Psi(\mathcal{U}_i)$ are both isomorphisms, which are (up to a non-zero scalar) inverse to each other. The natural transformation Φ is an isomorphism on a generating class of \mathcal{D}_0 . Thus, it is an isomorphism for all objects of \mathcal{D}_0 and \mathcal{D}_0 is Calabi-Yau of dimension r . ◀

Proposition 3.3.9 *Let X be a smooth projective variety, let $\mathcal{T} \subset D^b(X)$ be a full admissible subcategory which is Calabi-Yau of dimension 4 and let B a smooth algebraic variety. Let \mathcal{D} be a smooth deformation of \mathcal{T} over B with respect to $\pi : \mathcal{X} \rightarrow B$. Assume that $\mathcal{O}_{\mathcal{X}} \in \mathcal{D}$ and that for all $b \neq 0$, the category \mathcal{D}_b is compact hyper-Kähler of dimension 4. Then, the category $\mathcal{D}_0 = \mathcal{T}$ is compact hyper-Kähler of dimension 4.*

Proof :

► We already know that \mathcal{T} is smooth, compact, regular and Calabi-Yau of dimension

4. Since $\mathcal{O}_{\mathcal{X}} \in \mathcal{D}$, we have $\mathcal{O}_{\mathcal{X}_b} \subset \mathcal{D}_b$, for all $b \in B$, by B -linearity. Hence $H^\bullet(\mathcal{O}_{\mathcal{X}_b})$ is a homological unit for \mathcal{D}_b . For all $b \neq 0$, we know by hypothesis that $H^\bullet(\mathcal{O}_{\mathcal{X}_b}) = \mathbb{C}[t]/(t^3)$, with t in degree 2. By invariance of Hodge numbers in smooth family, we have $H^\bullet(\mathcal{O}_{\mathcal{X}_0}) = \mathbb{C}[t]/(t^3)$ as a graded vector space. But the category \mathcal{D}_0 is Calabi-Yau of dimension 4, hence the pairing:

$$H^2(\mathcal{O}_{\mathcal{X}_0}) \times H^2(\mathcal{O}_{\mathcal{X}_0}) \rightarrow H^4(\mathcal{O}_{\mathcal{X}_0}) \simeq \mathbb{C}$$

given by the Yoneda product coincide with the Serre-duality pairing : it is non degenerate. As $\dim H^2(\mathcal{O}_{\mathcal{X}_0}) = 1$, we find an isomorphism of graded algebras : $H^\bullet(\mathcal{O}_{\mathcal{X}_0}) = \mathbb{C}[t]/(t^3)$, with t in degree 2. This proves that \mathcal{D}_0 is compact hyper-Kähler. ◀

One would obviously like to generalize this result in higher dimension. However, the non-degeneracy of the Serre-duality pairing does not have so strong consequences in higher dimension as it has in dimension 4.

4 Partial conclusion

In [Abub], I will focus on a four-dimensional example of compact hyper-Kähler category. Its construction is based on a modular four-dimensional V-manifold exhibited by Markushevich and Tikhomirov [MT07]. I will prove that this category can not be the deformation of a commutative compact hyper-Kähler fourfold, thus showing that the moduli of non-commutative compact hyper-Kähler manifolds (if it exists) contains a component which is purely non-commutative. I will also discuss some connections with the theory of categorical fixed point loci and non-commutative compactifications of moduli spaces of Higgs bundles.

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